

Viscosity characterization of the explosion time distribution for diffusions

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We show that the tail distribution U of the explosion time for a multidimensional diffusion (and more generally, a suitable function \mathcal{U} of the Feynman-Kac type involving the explosion time) is a viscosity solution of an associated parabolic partial differential equation (PDE), provided that the dispersion and drift coefficients of the diffusion are continuous. This generalizes a result of Karatzas and Ruf (2015), who characterize U as a classical solution of a Cauchy problem for the PDE in the one-dimensional case, under the stronger condition of local Hölder continuity on the coefficients. We also extend their result to \mathcal{U} in the one-dimensional case by establishing the joint continuity of \mathcal{U} . Furthermore, we show that \mathcal{U} is dominated by any nonnegative classical supersolution of this Cauchy problem. Finally, we consider another notion of weak solvability, that of the distributional (sub/super)solution, and show that \mathcal{U} is no greater than any nonnegative distributional supersolution of the relevant PDE.

Keywords: distributional solution, explosion time, Feynman-Kac formula, minimal solution, multidimensional diffusion, second-order parabolic PDE, viscosity solution.

1. Notation and previous results

Let \mathcal{O} be a fixed domain of \mathbb{R}^n , not necessarily bounded, and $b = (b_1, \dots, b_n) : \mathcal{O} \rightarrow \mathbb{R}^n$ and $\sigma = (\sigma_{ik})_{n \times n} : \mathcal{O} \rightarrow \mathbb{M}(n)$ two measurable functions. We assume throughout this paper that for every $x \in \mathcal{O}$, the stochastic differential equation

$$dX_i(t) = \sum_k \sigma_{ik}(X(t)) dW_k(t) + b_i(X(t)) dt, \quad i = 1, \dots, n, \quad X(0) = x \quad (1.1)$$

admits a weak solution $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which is *unique* in law and defined up until the *explosion time*

$$S := \inf \{t \geq 0 : X(t) \notin \mathcal{O}\}.$$

Here $\mathbb{M}(n)$ is the set of $n \times n$ real matrices, $W(\cdot) = (W_1(\cdot), \dots, W_n(\cdot))$ is an n -dimensional Brownian motion, and the state process $X(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))$ is strongly Markovian as a consequence of the uniqueness in law. In (1.1) and throughout this paper, summations extend from 1 to n . We shall denote the distribution of the process $X(\cdot)$ starting at x by \mathbb{P}_x ; the expectation $\mathbb{E}^{\mathbb{P}_x}$ corresponding to this probability measure by \mathbb{E}_x ; and by U the *tail distribution function* of the explosion time S :

$$[0, \infty) \times \mathcal{O} \ni (t, x) \mapsto U(t, x) := \mathbb{P}_x[S > t] \in [0, 1]. \quad (1.2)$$

In [9], the authors studied the one-dimensional case $n = 1$, when $\mathcal{O} = (\ell, r)$ is a fixed open interval with $-\infty \leq \ell < r \leq \infty$ and $b : \mathcal{O} \rightarrow \mathbb{R}$ and $\sigma : \mathcal{O} \rightarrow \mathbb{R} \setminus \{0\}$ are two measurable functions satisfying

$$\int_K \left(\frac{1}{\sigma^2(y)} + \frac{|b(y)|}{\sigma^2(y)} \right) dy < \infty$$

for every compact set $K \subset \mathcal{O}$. According to the same arguments as in [2], [3] or [10, Theorem 5.5.15], the stochastic differential equation (1.1) admits then a weak solution defined up until the explosion time S , which is unique in law. As a result, the state process $X(\cdot)$ has the strong Markov property. It is then shown in [9] that, in the one-dimensional case and under the above conditions, the function U of (1.2) is jointly continuous, and furthermore, that if σ and b are locally Hölder continuous, then U solves the Cauchy problem for the parabolic partial differential equation (PDE)

$$u_t(t, x) - \frac{1}{2}\sigma^2(x)u_{xx}(t, x) - b(x)u_x(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathcal{O} \quad (1.3)$$

with the initial condition

$$u(0, x) = 1, \quad x \in \mathcal{O}, \quad (1.4)$$

and is in fact the smallest nonnegative (super)solution of the Cauchy problem (1.3), (1.4).

Remark: In the one-dimensional case, the Feller test provides necessary and sufficient conditions under which explosion occurs with positive probability, i.e., $\mathbb{P}_x[S < \infty] > 0$; in this case the Cauchy problem (1.3), (1.4) has lots of classical solutions in addition to its trivial solution $u \equiv 1$, of which the function U in (1.2) is the smallest. A host of examples is provided in [9, Section 6]. \square

A natural question to ask then, is whether U is still a solution to (1.3), probably in some weak or generalized sense, when we weaken the Hölder condition on σ and b to *continuity only*. The answer is affirmative. Moreover, this result can be generalized to multidimensional settings and to functions \mathcal{U} given by multidimensional *Feynman-Kac-type* expressions involving the explosion time, namely

$$\mathcal{U}(t, x) := \mathbb{E}_x \left[\mathbf{1}_{\{S > t\}} f(X(t)) \exp \left(- \int_0^t h(X(s)) ds \right) \right]. \quad (1.5)$$

Here f and $h : \mathcal{O} \rightarrow \mathbb{R}$ are measurable functions, such that $\mathcal{U}(t, x)$ is well-defined by (1.5) and finite for all $(t, x) \in (0, \infty) \times \mathcal{O}$; when $f \equiv 1$ and $h \equiv 0$, the function \mathcal{U} coincides with the tail distribution U of the explosion time, as defined in (1.2).

Remark: A sufficient condition for the right-hand side of (1.5) to be well-defined and finite, is that f be bounded and h be bounded from below (e.g., $h \geq 0$) on \mathcal{O} . \square

For ease of notation we introduce the continuous, adapted, strictly positive process

$$Y(t) := \exp \left(- \int_0^t h(X(s)) ds \right), \quad (1.6)$$

and thus $\mathcal{U}(t, x) = \mathbb{E}_x [\mathbf{1}_{\{S > t\}} f(X(t)) Y(t)]$. Note the dynamics $dY(t) = -h(X(t))Y(t) dt$.

1.1. Preview

Section 2 introduces the definition of *viscosity solution* and characterizes the function \mathcal{U} as a viscosity solution to the Feynman-Kac-type version of (1.3), (1.4), namely

$$(u_t - \mathcal{L}'u)(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathcal{O} \quad (1.7)$$

with

$$u(0, x) = f(x), \quad x \in \mathcal{O}, \quad (1.8)$$

where we have set

$$\mathcal{L}'u(t, x) := \frac{1}{2} \sum_{i,j} a_{ij}(x) u_{x_i x_j}(t, x) + \sum_i b_i(x) u_{x_i}(t, x) - h(x) u(t, x), \quad (a_{ij})_{n \times n} = a := \sigma \sigma^T. \quad (1.9)$$

Section 3 establishes the joint continuity of \mathcal{U} in its arguments (t, x) in the one-dimensional case, and therefore characterizes \mathcal{U} as a classical solution of the Cauchy problem (1.7), (1.8). Section 4 shows that \mathcal{U} is dominated by *any* nonnegative classical supersolution of this Cauchy problem. Finally, Section 5 defines another kind of weak solution, the *distributional (sub/super) solution*, and shows that \mathcal{U} is no greater than *any* nonnegative distributional supersolution of (1.7).

1.2. The special case U

In the case of $\mathcal{U} \equiv U$, our results can be summarized as follows.

Theorem 1.1. *Assume that for every $x \in \mathcal{O}$, the stochastic differential equation (1.1) admits a weak solution $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}(t)\}_{0 \leq t < \infty}$ which is unique in law and defined up until the explosion time $S := \inf\{t \geq 0 : X(t) \notin \mathcal{O}\}$. Denote the distribution of the process $X(\cdot)$ starting at x by \mathbb{P}_x . If the functions $a := \sigma \sigma^T$ and b are continuous, then the function $U(t, x) := \mathbb{P}_x[S > t]$, $(t, x) \in [0, \infty) \times \mathcal{O}$ is a viscosity solution of the following parabolic equation:*

$$(u_t - \mathcal{L}u)(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathcal{O} \quad (1.10)$$

with

$$\mathcal{L}u(t, x) := \frac{1}{2} \sum_{i,j} a_{ij}(x) u_{x_i x_j}(t, x) + \sum_i b_i(x) u_{x_i}(t, x), \quad (1.11)$$

and thus a viscosity solution of the Cauchy problem (1.10), (1.4), since $U(0, \cdot) = 1$.

Theorem 1.2. *The function U is dominated by any nonnegative, classical supersolution $\mathcal{U} \in C([0, \infty) \times \mathcal{O}) \cap C^{1,2}((0, \infty) \times \mathcal{O})$ of the Cauchy problem (1.10), (1.4).*

Theorem 1.3. *Assume that σ and b are locally bounded and a is locally strictly elliptic. If $v \in W_{loc}^{1,2}((0, \infty) \times \mathcal{O})$ is a nonnegative distributional supersolution of (1.10) with $v(0, \cdot) \geq 1$ on \mathcal{O} , then $v \geq U$ on $[0, \infty) \times \mathcal{O}$.*

Consequently, if the function U belongs to the space $W_{loc}^{1,2}((0, \infty) \times \mathcal{O})$ and is a distributional supersolution of (1.10), then it is the smallest nonnegative distributional supersolution of (1.10) that satisfies $u(0, \cdot) \geq 1$ on \mathcal{O} .

2. Viscosity characterization of $\mathcal{U}(t, x)$

This section develops a Feynman-Kac-type result involving the explosion time, which characterizes the function $(t, x) \mapsto \mathcal{U}(t, x)$ as a viscosity solution of the associated parabolic equation (1.7).

2.1. Definition of viscosity solutions

We first recall from [1] the definition of viscosity (sub/super)solutions of a second-order parabolic PDE. Let \mathcal{O} be an open subset of \mathbb{R}^n and $(t, x, y, p, q) \mapsto F(t, x, y, p, q)$ a continuous, real-valued mapping defined on $(0, \infty) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ and satisfying the ellipticity condition

$$F(t, x, y, p, q_1) \leq F(t, x, y, p, q_2) \quad \text{whenever } q_1 \geq q_2$$

for all $(t, x, y, p) \in (0, \infty) \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n$. Here $\mathbb{S}(n)$ is the set of $n \times n$ real symmetric matrices.

Consider the second-order parabolic PDE

$$u_t(t, x) + F(t, x, u(t, x), Du(t, x), D^2u(t, x)) = 0, \quad (t, x) \in (0, \infty) \times \mathcal{O} \quad (2.1)$$

with $Du = (u_{x_1}, u_{x_2}, \dots, u_{x_n})'$ and $D^2u = (u_{x_i x_j})_{n \times n}$.

Definition 2.1. (i) We say that a function $u : (0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}$ is a viscosity supersolution of the equation (2.1), if

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u_*(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \geq 0 \quad (2.2)$$

holds for all $(t_0, x_0) \in (0, \infty) \times \mathcal{O}$ and test functions $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$ such that (t_0, x_0) is a minimum of $u_* - \varphi$ on $(0, \infty) \times \mathcal{O}$. We have denoted by

$$u_*(t, x) := \liminf_{(s, y) \rightarrow (t, x)} u(s, y), \quad (t, x) \in (0, \infty) \times \mathcal{O}$$

the lower-semicontinuous envelope of u , i.e., the largest lower-semicontinuous function dominated pointwise by the function u .

(ii) Similarly, a function $u(t, x) : (0, \infty) \times \mathcal{O} \rightarrow \mathbb{R}$ is a viscosity subsolution of (2.1), if

$$\varphi_t(t_0, x_0) + F(t_0, x_0, u^*(t_0, x_0), D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \leq 0 \quad (2.3)$$

holds for all $(t_0, x_0) \in (0, \infty) \times \mathcal{O}$ and test functions $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$ such that (t_0, x_0) is a maximum of $u^* - \varphi$ on $(0, \infty) \times \mathcal{O}$. We have denoted by

$$u^*(t, x) := \limsup_{(s, y) \rightarrow (t, x)} u(s, y), \quad (t, x) \in (0, \infty) \times \mathcal{O} \quad (2.4)$$

the upper-semicontinuous envelope of u , i.e., the smallest upper-semicontinuous function that dominates pointwise the function u .

(iii) Finally, we say that u is a viscosity solution of (2.1), if it is both a viscosity supersolution and a viscosity subsolution of this equation.

In addition, this definition is not changed if the minimum and maximum are strict and/or local.

In our setting we have

$$F(t, x, y, p, q) = -\frac{1}{2} \sum_{i,j} a_{ij}(x) q_{ij} - \sum_i b_i(x) p_i + h(x) y,$$

and the left-hand sides of (2.2) and (2.3) simplify to $(\varphi_t - \mathcal{L}'\varphi)(t_0, x_0)$ with \mathcal{L}' defined in (1.9). Since $a = \sigma\sigma^T$ is positive-semidefinite, the function F satisfies the ellipticity condition. We also need F to be continuous, which means that the functions a_{ij} , b_i and h must be continuous for all indices i, j .

2.2. Viscosity characterization

Theorem 2.2. *Assume that the functions a , b and h are continuous. Then the function \mathcal{U} of (1.5) is a viscosity solution of the parabolic equation (1.7), and thus a viscosity solution of the Cauchy problem (1.7), (1.8), since $\mathcal{U}(0, x) = f(x)$.*

We first highlight the main idea for the proof of viscosity subsolution property without many of the technicalities. Similar arguments will lead to the viscosity supersolution property.

We prove by contradiction, assuming the contrary of (2.3) in Definition 2.1 that there exist $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$ and $(t_0, x_0) \in (0, \infty) \times \mathcal{O}$ such that (t_0, x_0) is a strict maximum of $\mathcal{U}^* - \varphi$, that the maximal value equals 0, and that

$$(\varphi_t - \mathcal{L}'\varphi)(t_0, x_0) > 0. \quad (2.5)$$

It then follows from the definition (2.4) of \mathcal{U}^* that we can take a pair (t^*, x^*) close to (t_0, x_0) such that $(\varphi - \mathcal{U})(t^*, x^*) (\geq 0)$ is sufficiently small, say less than a small constant C_2 . Under \mathbb{P}_{x^*} , we have

$$\varphi(t^*, x^*) - \mathbb{E}_{x^*} [\varphi(t^* - \nu, X(\nu))Y(\nu)] = \mathbb{E}_{x^*} \left[\int_0^\nu \mathcal{G}(t^* - t, X(t))Y(t) dt \right] \geq 0, \quad (2.6)$$

for any sufficiently small stopping time ν , where the process $Y(\cdot)$ is defined (1.6), and $\mathcal{G}(t, x) := (\varphi_t - \mathcal{L}'\varphi)(t, x)$, which is positive for any (t, x) sufficiently close to (t^*, x^*) by assumption (2.5).

On the other hand, in (2.6), we estimate $\varphi(t^*, x^*)$ from above by $C_2 + \mathcal{U}(t^*, x^*)$, and $\varphi(t^* - \nu, X(\nu))$ from below by $C_1 + \mathcal{U}(t^* - \nu, X(\nu))$ with C_1 a small positive constant, and deduce that

$$\mathcal{U}(t^*, x^*) > \mathbb{E}_{x^*} [\mathcal{U}(t^* - \nu, X(\nu))Y(\nu)].$$

This inequality contradicts the martingale property of the process $\mathcal{U}(t^* - \nu, X(\nu))Y(\nu)$, which is a consequence of the strong Markov property of $X(\cdot)$.

When implementing this idea, the stopping time ν needs to be not only small, but also satisfy that on $[0, \nu]$, the process $X(\cdot)$ is bounded and close to x^* ; however, the ν cannot be too small, in order to ensure that $\varphi(t^* - \nu, X(\nu)) \geq C_1 + \mathcal{U}(t^* - \nu, X(\nu))$ holds. These considerations inspire us to design ν as in (2.12) below. \square

Proof of Theorem 2.2. Let us show that \mathcal{U} is a viscosity subsolution to (1.7). The proof for the viscosity supersolution property is similar.

According to the definition of viscosity subsolution, it suffices to verify that for any test function $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$, and for any $(t_0, x_0) \in (0, \infty) \times \mathcal{O}$ with

$$(\mathcal{U}^* - \varphi)(t_0, x_0) = 0 > (\mathcal{U}^* - \varphi)(t, x), \quad \forall (t, x) \in (0, \infty) \times \mathcal{O}, \quad (2.7)$$

i.e., such that (t_0, x_0) is a strict maximum of $\mathcal{U}^* - \varphi$, we have the inequality

$$\mathcal{G}(t_0, x_0) \leq 0 \quad \text{for the function} \quad \mathcal{G}(t, x) := (\varphi_t - \mathcal{L}'\varphi)(t, x), \quad (2.8)$$

with \mathcal{L}' and \mathcal{U}^* defined in (1.9) and (2.4). We shall argue this by contradiction, assuming that

$$\mathcal{G}(t_0, x_0) > 0.$$

Since the function \mathcal{G} just introduced in (2.8) is continuous, there exists a neighborhood

$$\mathcal{N}_\delta := (t_0 - \delta, t_0 + \delta) \times B_\delta(x_0) \subset\subset (0, \infty) \times \mathcal{O} \quad (2.9)$$

of (t_0, x_0) , on which $\mathcal{G} > 0$ holds (“ $\subset\subset$ ” means *compactly contained in*). Recalling (2.7) and the condition that h is continuous and thus locally bounded, we then introduce the constants

$$C_1 := -\max_{\partial\mathcal{N}_\delta} (\mathcal{U}^* - \varphi)(t, x) > 0, \quad C_2 := C_1 \exp \left(-2\delta \sup_{x \in B_\delta(x_0)} |h(x)| \right). \quad (2.10)$$

We observe that

$$\limsup_{(t,x) \rightarrow (t_0, x_0)} (\mathcal{U} - \varphi)(t, x) = (\mathcal{U}^* - \varphi)(t_0, x_0) = 0$$

holds by the definition (2.4) of \mathcal{U}^* and the continuity of φ , hence there exists $(t^*, x^*) \in \mathcal{N}_\delta$ such that

$$(\mathcal{U} - \varphi)(t^*, x^*) > -C_2. \quad (2.11)$$

Let us consider the stopping time

$$\nu(\omega) := \inf \{s \in (0, t^*] : (t^* - s, X(s)) \notin \mathcal{N}_\delta\} \wedge t^*, \quad (2.12)$$

and note that the definition (2.9) of \mathcal{N}_δ implies

$$\nu < S \quad \text{and} \quad \nu \leq t^* - (t_0 - \delta) = (t^* - t_0) + \delta < t^* \wedge (2\delta). \quad (2.13)$$

Now thanks to the assumption $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$, we can apply Itô's change of variable rule to $\varphi(t^* - t, X(t))Y(t)$ for $t \in [0, \nu] \subset [0, t^*)$ and plug in (1.1) to derive the semimartingale decomposition

$$d[\varphi(t^* - t, X(t))Y(t)] = -\mathcal{G}(t^* - t, X(t))Y(t) dt + \sum_{i,k} \varphi_{x_i}(t^* - t, X(t)) \sigma_{ik}(X(t))Y(t) dW_k(t), \quad (2.14)$$

with the process $Y(\cdot)$ defined in (1.6) and the function \mathcal{G} in (2.8) (see Appendix B for a proof).

Integrating (2.14) with respect to t over $[0, \nu]$ and taking the expectation under \mathbb{P}_{x^*} yields

$$\varphi(t^*, x^*) - \mathbb{E}_{x^*} [\varphi(t^* - \nu, X(\nu))Y(\nu)] = \mathbb{E}_{x^*} \left[\int_0^\nu \mathcal{G}(t^* - t, X(t)) Y(t) dt \right] \geq 0. \quad (2.15)$$

Here the last inequality comes from the assumption $\mathcal{G} > 0$ on \mathcal{N}_δ , whereas the equality holds because the expectations of the integrals with respect to $dW_k(t)$ have all vanished, due to:

- (1) the uniform boundedness of $Y(t)$ on $[0, \nu]$; in fact, we have $Y(t) \leq \exp \left(t \sup_{x \in B_\delta(x_0)} |h(x)| \right) \leq \exp \left(2\delta \sup_{x \in B_\delta(x_0)} |h(x)| \right)$ for all $0 \leq t \leq \nu < 2\delta$; and
- (2) the boundedness of φ_{x_i} on \mathcal{N}_δ , and a_{ij} and thus σ_{ik} on $B_\delta(x_0)$ (recalling that $a = \sigma\sigma^T$).

Combining (2.15) with (2.10) and (2.11) leads to

$$\begin{aligned} 0 &\leq \varphi(t^*, x^*) - \mathbb{E}_{x^*} [\varphi(t^* - \nu, X(\nu))Y(\nu)] < C_2 + \mathcal{U}(t^*, x^*) - \mathbb{E}_{x^*} [(C_1 + \mathcal{U}(t^* - \nu, X(\nu)))Y(\nu)] \\ &= \{C_2 - C_1 \mathbb{E}_{x^*} [Y(\nu)]\} + \{\mathcal{U}(t^*, x^*) - \mathbb{E}_{x^*} [\mathcal{U}(t^* - \nu, X(\nu))Y(\nu)]\}. \end{aligned} \quad (2.16)$$

We see that the first term on the right-hand side is at most zero by the definition (2.10) of C_2 and the definition (1.6) of $Y(\cdot)$, along with the fact $\nu < 2\delta$ from (2.13). Thus we will arrive at a contradiction as soon as we have shown that the second term on the right-hand side of (2.16) equals zero, namely

$$\mathbb{E}_{x^*} [\mathcal{U}(t^* - \nu, X(\nu))Y(\nu)] = \mathcal{U}(t^*, x^*). \quad (2.17)$$

In fact, it follows from the strong Markov property of $X(\cdot)$ that

$$\begin{aligned}\mathcal{U}(t^* - \nu, X(\nu)) &= \mathbb{E}_\xi \left[\mathbf{1}_{\{S > t^* - \nu\}} f(X(t^* - \nu)) \exp \left(- \int_0^{t^* - \nu} h(X(s)) ds \right) \right] \Big|_{\xi = X(\nu)} \\ &= \mathbb{E}_{x^*} \left[\mathbf{1}_{\{S > t^*\}} f(X(t^*)) \exp \left(- \int_\nu^{t^*} h(X(s)) ds \right) \right] \Big| \mathcal{F}(\nu), \quad \mathbb{P}_{x^*}\text{-a.s.},\end{aligned}$$

thus plugging into the left-hand side of (2.17) along with the definition (1.6) of $Y(\cdot)$ yields

$$\begin{aligned}& \mathbb{E}_{x^*} \left[\mathbb{E}_{x^*} \left[\mathbf{1}_{\{S > t^*\}} f(X(t^*)) \exp \left(- \int_\nu^{t^*} h(X(s)) ds \right) \exp \left(- \int_0^\nu h(X(s)) ds \right) \right] \Big| \mathcal{F}(\nu) \right] \\ &= \mathbb{E}_{x^*} \left[\mathbf{1}_{\{S > t^*\}} f(X(t^*)) \exp \left(- \int_0^{t^*} h(X(s)) ds \right) \right] = \mathcal{U}(t^*, x^*). \quad \square\end{aligned}$$

3. Joint continuity of \mathcal{U} in the one-dimensional case

In this section, we fix $n = 1$. The function $U(t, x)$ is shown to be jointly continuous in (t, x) on $[0, \infty) \times \mathcal{O}$ [9, Proposition 4.3]. This section generalizes this result for $\mathcal{U}(t, x)$ by starting with the continuity in t and then using a “coupling” argument in conjunction with [9, (4.4)].

Thanks to the joint continuity of $\mathcal{U}(t, x)$, arguments similar to those in [9, Lemma 5.1 and Proposition 5.2] show that \mathcal{U} is actually a classical solution of the Cauchy problem (1.7), (1.8), whenever the function f is continuous and functions $a > 0$, b and $h \geq 0$ are locally Hölder continuous. This result extends [9, Proposition 5.2] to the more general Feynman-Kac context (see Corollary 3.3).

Theorem 3.1. *When $n = 1$, the function $\mathcal{U}(t, x)$ is jointly continuous in (t, x) on $[0, \infty) \times \mathcal{O}$ if f and h are bounded, $a > 0$ and either of the following conditions holds:*

- (i) *f is Hölder continuous and a and b are bounded;*
- (ii) *$f \in C^2(\mathcal{O})$ and the functions $f'\sigma$ and $f'b + \frac{1}{2}f''a$ are bounded on \mathcal{O} .*

Remark 3.2. The tail distribution function $U(t, x)$ with $f \equiv 1$ and $h \equiv 0$ is a special case that satisfies Condition (ii).

Corollary 3.3. *Under the assumptions of Theorem 3.1, if the functions a , b and $h \geq 0$ are locally Hölder continuous, then $\mathcal{U}(t, x)$ is a classical solution of the Cauchy problem (1.7), (1.8).*

Proof of Theorem 3.1. Let $C_0 > 0$ be an upper bound of $|f|$ and $|h|$ on \mathcal{O} , then

$$0 < Y(t) \leq e^{tC_0}, \quad \forall t \in [0, \infty). \quad (3.1)$$

We first prove the continuity of $\mathcal{U}(t, x)$ in t .

Lemma 3.4. *The function $t \mapsto \mathcal{U}(t, x)$ is continuous for any given $x \in \mathcal{O}$.*

Proof. Fix $(t, x) \in [0, \infty) \times \mathcal{O}$. Let us show the left-continuity. Switching t and t' below will give a proof for the right-continuity. Assume that $t \in (0, \infty)$. For any $t' \in [0, t)$, by definition we have

$$\begin{aligned} \mathcal{U}(t', x) - \mathcal{U}(t, x) &= \mathbb{E}_x [\mathbf{1}_{\{S > t'\}} f(X(t')) Y(t')] - \mathbb{E}_x [\mathbf{1}_{\{S > t\}} f(X(t)) Y(t)] \\ &= \mathbb{E}_x [(\mathbf{1}_{\{S > t'\}} - \mathbf{1}_{\{S > t\}}) f(X(t')) Y(t')] + \mathbb{E}_x [\mathbf{1}_{\{S > t\}} f(X(t')) (Y(t') - Y(t))] \\ &\quad + \mathbb{E}_x [\mathbf{1}_{\{S > t\}} (f(X(t')) - f(X(t))) Y(t)] \\ &=: \Delta_1 + \Delta_2 + \Delta_3. \end{aligned}$$

It then suffices to show that each $\Delta_j \rightarrow 0$ as $t' \uparrow t$.

Case 1. $j = 1$. Since $\mathbf{1}_{\{S > t\}}$ is decreasing in t and U is continuous, we obtain

$$|\Delta_1| \leq C_0 e^{t C_0} \mathbb{E}_x [\mathbf{1}_{\{S > t'\}} - \mathbf{1}_{\{S > t\}}] = C_0 e^{t C_0} [U(t', x) - U(t, x)] \rightarrow 0, \quad \text{as } t' \uparrow t.$$

Case 2. $j = 2$. For any $t < S$, we have

$$|Y(t') - Y(t)| = \left| Y(t) \left[\exp \left(\int_{t'}^t h(X(s)) ds \right) - 1 \right] \right| \leq e^{t C_0} (e^{(t-t') C_0} - 1),$$

since

$$e^{(t-t') C_0} - 1 \geq \exp \left(\int_{t'}^t h(X(s)) ds \right) - 1 \geq e^{-(t-t') C_0} - 1 \geq - (e^{(t-t') C_0} - 1)$$

(as $z + 1/z \geq 2$ for all $z > 0$). Hence

$$|\Delta_2| \leq C_0 \mathbb{E}_x [\mathbf{1}_{\{S > t\}} |Y(t') - Y(t)|] \leq C_0 e^{t C_0} (e^{(t-t') C_0} - 1) \rightarrow 0, \quad \text{as } t' \uparrow t.$$

Case 3. $j = 3$. (i) If Condition (i) holds, there exist constants $\alpha \in (0, 1]$ and $C_\alpha, C > 0$ such that $|f(x) - f(y)| \leq C_\alpha |x - y|^\alpha$ and $a(x), |b(x)| \leq C$ for all $x, y \in \mathcal{O}$. Thus Hölder's inequality gives

$$|\Delta_3| \leq e^{t C_0} \mathbb{E}_x [\mathbf{1}_{\{S > t\}} |f(X(t)) - f(X(t'))|] \leq e^{t C_0} C_\alpha \mathbb{E}_x [|\Xi(t')|^\alpha] \leq e^{t C_0} C_\alpha (\mathbb{E}_x [|\Xi(t')|^2])^{\alpha/2}, \quad (3.2)$$

where $\Xi(s) := \mathbf{1}_{\{S > t\}} (X(t) - X(s \wedge t))$, $s \in [0, \infty)$. Let us recall from (1.1) that

$$\Xi(s) = \mathbf{1}_{\{S > t\}} \int_s^t \sigma(X(\varsigma)) dW(\varsigma) + \mathbf{1}_{\{S > t\}} \int_s^t b(X(\varsigma)) d\varsigma =: M(s) + A(s), \quad \forall s \leq t,$$

therefore Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequalities give

$$\mathbb{E}_x [|\Xi(t')|^2] \leq 2 \mathbb{E}_x [|M(t')|^2] + 2 \mathbb{E}_x [|A(t')|^2] \leq 8 \mathbb{E}_x [\langle M \rangle(t')] + 2 \mathbb{E}_x [|A(t')|^2]. \quad (3.3)$$

Since $\sigma^2 = a \leq C$ and $|b| \leq C$ on \mathcal{O} , we obtain

$$\langle M \rangle(t') = \mathbf{1}_{\{S > t\}} \int_{t'}^t \sigma^2(X(\varsigma)) d\varsigma \leq (t - t') C \quad \text{and} \quad |A(t')| \leq (t - t') C. \quad (3.4)$$

Substituting into (3.3) and then into (3.2) yields

$$|\Delta_3| \leq e^{t C_0} C_\alpha [8(t - t') C + 2(t - t')^2 C^2]^{\alpha/2} \rightarrow 0, \quad \text{as } t' \uparrow t. \quad (3.5)$$

(ii) If Condition (ii) holds, let $C' > 0$ be an upper bound of $(f'\sigma)^2$ and $|f'b + \frac{1}{2}f''a|$ on \mathcal{O} . Our idea is similar to that of (i) with the process $\Xi(s)$ replaced by $\Xi'(s) := \mathbf{1}_{\{S>t\}}(f(X(t)) - f(X(s \wedge t)))$, $s \in [0, \infty)$, the exponent α set to 1 and the constant C_α removed. More precisely, we have

$$|\Delta_3| \leq e^{tC_0} \mathbb{E}_x[\mathbf{1}_{\{S>t\}}|f(X(t)) - f(X(t'))|] = e^{tC_0} \mathbb{E}_x[|\Xi'(t')|] \leq e^{tC_0} (\mathbb{E}_x[|\Xi'(t')|^2])^{1/2}.$$

For any $0 \leq s \leq t < S$, we apply Itô's change of variable rule to $f(X(\varsigma))$, $\varsigma \in [s, t]$ and plug in (1.1) to get

$$\Xi'(s) = \mathbf{1}_{\{S>t\}} \int_s^t (f'\sigma)(X(\varsigma)) dW(\varsigma) + \mathbf{1}_{\{S>t\}} \int_s^t \left(f'b + \frac{1}{2}f''a\right)(X(\varsigma)) d\varsigma =: M'(s) + A'(s).$$

Thus (3.3)–(3.5) hold for Ξ', M', A' and C' , with σ replaced by $f'\sigma$ in (3.4), C_α removed and $\alpha = 1$.

Switching t and t' in the above argument proves the right-continuity. More precisely, we summarize the above estimates as the following inequality that holds for all $t, t' \in [0, \infty)$ to complete the proof:

$$\begin{aligned} |\mathcal{U}(t', x) - \mathcal{U}(t, x)| &\leq e^{(t \vee t')C_0} \left\{ C_0 |U(t', x) - U(t, x)| + C_0 (e^{|t-t'|C_0} - 1) \right. \\ &\quad \left. + \left[C_\alpha (8|t-t'|C + 2(t-t')^2C^2)^{\alpha/2} \right] \vee \left[(8|t-t'|C' + 2(t-t')^2C'^2)^{1/2} \right] \right\} \\ &=: e^{(t \vee t')C_0} C_0 |U(t', x') - U(0, x')| + \gamma(t, t') \rightarrow 0, \quad \text{as } t' \rightarrow t. \end{aligned} \quad (3.6) \quad \square$$

Now let us go back to the joint continuity. Define the stopping time

$$H_x := \inf \{t \geq 0 : X(t) = x\}, \quad x \in \mathcal{O}.$$

Fix $(t, x) \in [0, \infty) \times \mathcal{O}$. If $t = 0$, then for any $(t', x') \in [0, \infty) \times \mathcal{O}$, it follows from (3.6) that

$$\begin{aligned} |\mathcal{U}(t', x') - \mathcal{U}(0, x)| &\leq |\mathcal{U}(t', x') - \mathcal{U}(0, x')| + |\mathcal{U}(0, x') - \mathcal{U}(0, x)| \\ &\leq e^{t'C_0} C_0 |U(t', x') - U(0, x')| + \gamma(0, t') + |f(x') - f(x)| \end{aligned}$$

Since f is continuous under either Condition (i) or (ii), $\gamma(0, t') \rightarrow 0$ as $t' \rightarrow 0$, and

$$|U(t', x') - U(0, x')| \leq |U(t', x') - U(0, x)| + |U(0, x) - U(0, x')| \rightarrow 0, \quad \text{as } (t', x') \rightarrow (0, x)$$

by the joint continuity of U , we conclude that $|\mathcal{U}(t', x') - \mathcal{U}(0, x)| \rightarrow 0$, as $(t', x') \rightarrow (0, x)$.

Now assume $t > 0$. For any $\varepsilon > 0$, according to Lemma 3.4, there exists $\delta \in (0, t/2)$ such that

$$|\mathcal{U}(\theta, x) - \mathcal{U}(t, x)| < \varepsilon, \quad \forall \theta \in (t - 2\delta, t + 2\delta). \quad (3.7)$$

For any $t' \in (t - \delta, t + \delta)$ (thus $t' > \delta$), we decompose $\mathcal{U}(t, x)$ and $\mathcal{U}(t', x')$ into two terms, respectively:

$$\mathcal{U}(t, x) = \mathbb{E}_{x'}[\mathbf{1}_{\{H_x < \delta\}} \mathcal{U}(t, x)] + \mathbb{E}_{x'}[\mathbf{1}_{\{H_x \geq \delta\}} \mathcal{U}(t, x)] =: E_1 + E_2,$$

$$\mathcal{U}(t', x') = \mathbb{E}_{x'}[\mathbf{1}_{\{S>t'\}} \mathbf{1}_{\{H_x < \delta\}} f(X(t'))Y(t')] + \mathbb{E}_{x'}[\mathbf{1}_{\{S>t'\}} \mathbf{1}_{\{H_x \geq \delta\}} f(X(t'))Y(t')] =: E'_1 + E'_2.$$

Then using the well-known property that $\lim_{x' \rightarrow x} \mathbb{P}_{x'}[H_x \geq \delta] = 0$ (a simple proof of this property can be found in [9, (4.4)]; see also [8, Section 3.3]), we obtain

$$|E_2| = |\mathcal{U}(t, x)| \mathbb{P}_{x'}[H_x \geq \delta] \rightarrow 0, \quad |E'_2| \leq C_0 e^{(t+\delta)C_0} \mathbb{P}_{x'}[H_x \geq \delta] \rightarrow 0, \quad \text{as } x' \rightarrow x.$$

Finally, the strong Markov property of $X(\cdot)$ implies that

$$\begin{aligned} E'_1 &= \mathbb{E}_{x'} [\mathbf{1}_{\{H_x < \delta\}} \mathbb{E}_{x'} [\mathbf{1}_{\{S > t'\}} f(X(t')) Y(t') \mid \mathcal{F}(H_x)]] \\ &= \mathbb{E}_{x'} [\mathbf{1}_{\{H_x < \delta\}} \mathbb{E}_x [\mathbf{1}_{\{S > t' - H_x\}} f(X(t' - H_x)) Y(t' - H_x)]] \\ &= \mathbb{E}_{x'} [\mathbf{1}_{\{H_x < \delta\}} \mathcal{U}(t' - H_x, x)]. \end{aligned}$$

Since $t' - H_x \in (t - 2\delta, t + 2\delta)$ holds on $\{H_x < \delta\}$, on the strength of (3.7) we arrive at

$$|E'_1 - E_1| = \mathbb{E}_{x'} [\mathbf{1}_{\{H_x < \delta\}} (\mathcal{U}(t' - H_x, x) - \mathcal{U}(t, x))] \leq \mathbb{E}_{x'} [\mathbf{1}_{\{H_x < \delta\}} |\mathcal{U}(t' - H_x, x) - \mathcal{U}(t, x)|] < \varepsilon.$$

Hence $\mathcal{U}(\cdot, \cdot)$ is jointly continuous at (t, x) . \square

Remark 3.5. We have a simpler proof for the right-continuity of $\mathcal{U}(t, x)$ in t , by taking advantage of the right-continuity of $\mathbf{1}_{\{S > \cdot\}}$ in conjunction with the continuity and boundedness of the paths of $f(X(\cdot))$ and $Y(\cdot)$ (see (3.1)). It follows from the dominated convergence theorem that

$$\lim_{t' \downarrow t} \mathcal{U}(t', x) = \lim_{t' \downarrow t} \mathbb{E}_x [\mathbf{1}_{\{S > t'\}} f(X(t')) Y(t')] = \mathbb{E}_x \left[\lim_{t' \downarrow t} \mathbf{1}_{\{S > t'\}} f(X(t')) Y(t') \right] = \mathcal{U}(t, x). \quad \square$$

4. Minimality

For the one-dimensional case, the function U of (1.2) is dominated by every nonnegative classical supersolution of the Cauchy problem (1.10), (1.4) [9, Proposition 5.3]. Therefore, whenever the function U is known in advance to be a classical solution of this problem (for instance, when the functions a and b are Hölder continuous), it is also the smallest nonnegative classical supersolution of this problem. This section proves these results in the more general Feynman-Kac case. Our minimality results (Theorem 4.1 and Corollary 4.2 below) have similarities – at least in spirit – to those in Proposition 5.3 of [9], Problem 3.5.1 of [13], in Exercise 4.4.7 of [10], as well as to [4], [5] and [15].

Theorem 4.1. *Assume that the function f is nonnegative on \mathcal{O} . Then the function \mathcal{U} defined in (1.5) is dominated by every nonnegative, classical supersolution $\mathcal{U}(t, x) \in C([0, \infty) \times \mathcal{O}) \cap C^{1,2}((0, \infty) \times \mathcal{O})$ of the Cauchy problem (1.7), (1.8).*

Combining Theorem 4.1 with Corollary 3.3 leads directly to the following corollary.

Corollary 4.2. *If the assumptions of Theorem 4.1 and Corollary 3.3 hold, then \mathcal{U} is the smallest nonnegative classical (super)solution of the Cauchy problem (1.7), (1.8).*

Proof of Theorem 4.1. It follows from definition (1.5) and the supersolution property that $\mathcal{U}(0, x) = f(x) \leq \mathcal{U}(0, x)$ holds on \mathcal{O} . Let $\mathcal{O}_m \subset \subset \mathcal{O}$ (i.e., $\overline{\mathcal{O}_m} \subset \mathcal{O}$) be a sequence of bounded domains in \mathbb{R}^n such that $\mathcal{O}_m \uparrow \mathcal{O}$ and define the stopping times

$$S_m := \inf \{t \geq 0 : X(t) \notin \mathcal{O}_m\}, \quad m \in \mathbb{N}. \quad (4.1)$$

Then $S_m \uparrow S$.

Let us fix a positive integer m . For any given $(T, x) \in (0, \infty) \times \mathcal{O}_m$, thanks to the assumption $\mathcal{U} \in C^{1,2}((0, \infty) \times \mathcal{O})$, we also have (2.14) with φ replaced by \mathcal{U} , t^* by T , and t by $t \wedge S_m$, $0 \leq t \leq T$. This way we deduce from the supersolution property that $\mathcal{U}_t - \mathcal{L}'\mathcal{U} \geq 0$ holds on

$(0, \infty) \times \mathcal{O}$, therefore the process $\mathcal{U}(T - t \wedge S_m, X(t \wedge S_m))Y(t \wedge S_m)$, $0 \leq t \leq T$ is a local \mathbb{P}_x -supermartingale.

This local supermartingale is nonnegative and hence a true supermartingale. Therefore we obtain

$$\begin{aligned} \mathcal{U}(T, x) &\geq \mathbb{E}_x[\mathcal{U}(T - T \wedge S_m, X(T \wedge S_m))Y(T \wedge S_m)] \geq \mathbb{E}_x[\mathbf{1}_{\{S_m > T\}} \mathcal{U}(0, X(T))Y(T)] \\ &\geq \mathbb{E}_x[\mathbf{1}_{\{S_m > T\}} f(X(T))Y(T)] \end{aligned}$$

by optional sampling. The last expression converges to $\mathbb{E}_x[\mathbf{1}_{\{S > T\}} f(X(T))Y(T)] = \mathcal{U}(T, x)$ as $m \rightarrow \infty$ by the monotone convergence theorem, so we conclude $\mathcal{U}(T, x) \geq \mathcal{U}(T, x)$. \square

5. The distributional solution

In this section we define and study yet another kind of weak solution, the *distributional solution*. We shall adopt the following definition from [11]. Set

$$\|f\|_{C(\overline{Q})} := \sup_{y \in \overline{Q}} |f(y)|, \quad \|f\|_{W^{1,2}(\overline{Q})} := \|f_t\|_{L^{n+1}(\overline{Q})} + \sum_i \|f_{x_i}\|_{L^{n+1}(\overline{Q})} + \sum_{i,j} \|f_{x_i x_j}\|_{L^{n+1}(\overline{Q})} + \|f\|_{C(\overline{Q})}.$$

Definition 5.1. ([11, Definition 2.1.1]) *Let Q be a bounded domain of \mathbb{R}^{n+1} . We denote by $W^{1,2}(Q)$ the space of functions $u : \overline{Q} \rightarrow \mathbb{R}$, for each of which there exists a sequence of functions $u^{(m)} \in C^{1,2}(\overline{Q})$ such that $\|u - u^{(m)}\|_{C(\overline{Q})} \rightarrow 0$ and $\|u^{(m)} - u^{(m')}\|_{W^{1,2}(\overline{Q})} \rightarrow 0$ as $m, m' \rightarrow \infty$.*

The continuity of the functions $u^{(m)}$ implies that $u \in C(\overline{Q})$. In addition, every function $u \in W^{1,2}(Q)$ possesses generalized (a.k.a. “weak”) derivatives u_t , u_{x_i} and $u_{x_i x_j}$ on Q , which are unique almost everywhere ([11, Definition 2.1.2]).

We generalize Definition 5.1 for unbounded domains as follows.

Definition 5.2. *Let Q' be a domain of \mathbb{R}^{n+1} , not necessarily bounded. Define*

$$W_{loc}^{1,2}(Q') := \{u : Q' \rightarrow \mathbb{R} \mid u \in W^{1,2}(Q) \text{ for all bounded domain } Q \subset Q'\}.$$

For $u \in W_{loc}^{1,2}(Q')$, we have $u \in C(\overline{Q})$ for all bounded domain $Q \subset Q'$, hence $u \in C(\overline{Q'})$ as well. Further, the generalized derivatives u_t , u_{x_i} and $u_{x_i x_j}$ can be shown to be well-defined and unique a.e. To see this, take a sequence of bounded domains $Q_m \subset Q'$ such that $Q_m \uparrow Q'$ and use the fact that the generalized derivatives are uniquely defined a.e. on each Q_m .

Now we are able to define the distributional (sub/super)solution. Let Q' be a domain of \mathbb{R}^{n+1} and $F(z, y, p, q)$ a continuous map from $Q' \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}(n)$ to \mathbb{R} satisfying the ellipticity condition:

$$F(z, y, p, q_1) \leq F(z, y, p, q_2) \quad \text{whenever } q_1 \geq q_2,$$

for all $(z, y, p) \in Q' \times \mathbb{R} \times \mathbb{R}^n$ (by analogy with the F in Section 2.1).

Consider a second-order parabolic PDE

$$u_t(t, x) + F((t, x), u(t, x), Du(t, x), D^2 u(t, x)) = 0, \quad (t, x) \in Q'. \quad (5.1)$$

Definition 5.3. We say that $u \in W_{loc}^{1,2}(Q')$ is a distributional sub(super)-solution of (5.1) if the generalized derivatives u_t , Du and D^2u can be chosen such that

$$u_t(t, x) + F((t, x), u(t, x), Du(t, x), D^2u(t, x)) \leq 0 \quad (\geq 0), \quad \forall (t, x) \in \Gamma \quad (5.2)$$

holds for some $\Gamma \subset Q'$ with $\text{meas}(Q' \setminus \Gamma) = 0$.

A distributional solution is a function that is both a distributional subsolution and a distributional supersolution.

In our setting, we have $Q' = (0, \infty) \times \mathcal{O}$ and

$$F((t, x), y, p, q) = -\frac{1}{2} \sum_{i,j} a_{ij}(x) q_{ij} - \sum_i b_i(x) p_i + h(x) y,$$

so the left-hand side of (5.2) simplifies to $(u_t - \mathcal{L}'u)(t, x)$ (recall the operator \mathcal{L}' from (1.9)). We see that F satisfies the ellipticity condition since $a = \sigma\sigma^T$ is positive-semidefinite.

5.1. Domination by nonnegative distributional supersolutions

We have the following minimality result for $\mathcal{U}(t, x)$.

Theorem 5.4. Assume that σ , b and h are locally bounded and a is locally strictly elliptic on \mathcal{O} . If $v \in W_{loc}^{1,2}((0, \infty) \times \mathcal{O})$ is a nonnegative distributional supersolution of (1.7) with

$$v(0, x) \geq f(x), \quad x \in \mathcal{O}, \quad (5.3)$$

then $v \geq \mathcal{U}$ on $[0, \infty) \times \mathcal{O}$.

Thus, if the function \mathcal{U} belongs to the space $W_{loc}^{1,2}((0, \infty) \times \mathcal{O})$ and is a distributional supersolution of (1.7), then it is the smallest nonnegative distributional supersolution of (1.7) that satisfies (5.3).

Proof. Set the generalized derivatives v_t , v_{x_i} and $v_{x_i x_j}$ to be identically equal to zero on $((0, \infty) \times \mathcal{O}) \setminus \Gamma$ (with Γ as defined in Definition 5.3). Fix an arbitrary point $x_0 \in \mathcal{O}$ and real number $T > 0$. Denote $Q_m = (0, T) \times \mathcal{O}_m$, with \mathcal{O}_m as in the proof of Theorem 4.1.

Let $u(t, x) = v(T - t, x)$, $(t, x) \in [0, T] \times \mathcal{O}$, then $u \in W^{1,2}(Q_m)$ is a distributional subsolution of $(u_t + \mathcal{L}'u)(t, x) = 0$ on Q_m . The following result is well known.

Lemma 5.5. ([11, Theorem 2.10.1]) Let Q be a bounded domain of \mathbb{R}^{n+1} and $u \in W^{1,2}(Q)$. For any $(s_0, x_0) \in Q$ and any stopping time $\tau \leq \tau(Q, s_0) := \inf \{t > 0 : (s_0 + t, X(t)) \notin Q\}$, we have

$$u(s_0, x_0) = \mathbb{E}_{x_0} \left[Y(\tau) u(s_0 + \tau, X(\tau)) - \int_0^\tau Y(t) (u_t + \mathcal{L}'u)(s_0 + t, X(t)) dt \right],$$

where $X(\cdot)$ is the diffusion of (1.1) starting at $X(0) = x_0$.

Applying the above lemma with $Q = Q_m$, $s_0 = 1/i$ and $\tau = \tau_{i,m} := (T - (1/i)) \wedge S_m = \tau(Q, s_0)$ with S_m defined in (4.1), and combining with the distributional subsolution property of u leads to

$$u\left(\frac{1}{i}, x_0\right) \geq \mathbb{E}_{x_0} \left[Y(\tau_{i,m}) u\left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m})\right) \right]. \quad (5.4)$$

Recall that v is continuous (see the explanation right below Definition 5.2), and therefore so is u . Thus u is bounded on $[0, T] \times \mathcal{O}$. Moreover, given m , for all $t \in [0, S_m \wedge T]$ we have

$$Y(t) = \exp \left(- \int_0^t h(X(s)) ds \right) \leq \exp \left(t \sup_{x \in \mathcal{O}_m} |h(x)| \right) \leq \exp \left(T \sup_{x \in \mathcal{O}_m} h(x) \right) < \infty,$$

since h is bounded on \mathcal{O}_m . Therefore, the family of random variables $\{Y(t), t \in [0, S_m \wedge T]\}$ is uniformly bounded. Notice further that the paths of the processes $X(\cdot)$ and $Y(\cdot)$ are continuous, and that $\lim_{i \rightarrow \infty} \tau_{i,m} = S_m \wedge T$. Taking $i \rightarrow \infty$ in (5.4) yields

$$\begin{aligned} v(T, x_0) &= u(0, x_0) = \lim_{i \rightarrow \infty} u(1/i, x_0) \geq \lim_{i \rightarrow \infty} \mathbb{E}_{x_0} \left[Y(\tau_{i,m}) u \left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m}) \right) \right] \\ &= \mathbb{E}_{x_0} \left[\lim_{i \rightarrow \infty} Y(\tau_{i,m}) u \left(\frac{1}{i} + \tau_{i,m}, X(\tau_{i,m}) \right) \right] = \mathbb{E}_{x_0} [Y(S_m \wedge T) v(T - S_m \wedge T, X(S_m \wedge T))] \\ &\geq \mathbb{E}_{x_0} [\mathbf{1}_{\{S_m > T\}} Y(T) f^+(X(T))] \rightarrow \mathbb{E}_{x_0} [\mathbf{1}_{\{S > T\}} Y(T) f^+(X(T))] \geq \mathcal{U}(T, x_0), \quad \text{as } m \rightarrow \infty, \end{aligned}$$

by the monotone convergence theorem, as claimed, where $f^+(\cdot) = f(\cdot) \vee 0$. \square

Remark 5.6. In the case of \mathcal{O} bounded with $\partial\mathcal{O}$ piecewise smooth, besides the condition of Corollary 3.3, we have another sufficient condition for \mathcal{U} to be a classical solution of (1.7), (1.8) whenever it can be checked directly that $\mathcal{U} \equiv 0$ on $(0, \infty) \times \partial\mathcal{O}$: the functions a , b , f and h are bounded and a is continuously differentiable and uniformly elliptic on \mathcal{O} . Under these conditions, a classical PDE result (see e.g. [12, Theorem III.12.1]) states that the Cauchy problem (1.7), (1.8) has a unique weak solution \mathcal{V} (in the distributional sense) that vanishes on $(0, \infty) \times \partial\mathcal{O}$; further, we have $\mathcal{V} \in C([0, T] \times \mathcal{O}) \cap C^{1,2}((0, T) \times \mathcal{O})$ for any $T \in (0, \infty)$. One can then show that \mathcal{V} coincides with \mathcal{U} on $(0, \infty) \times \mathcal{O}$ by integrating (2.14) with respect to t over $[0, S \wedge t^*]$, for any $t^* \in (0, \infty)$, and the conclusion follows. The result of $\mathcal{V} = \mathcal{U}$ is similar to Theorem II.2.3 and the discussion in [6, pp. 133–135]. The special case U was dealt with in [14, Theorem 2.7]. \square

5.2. Connections between distributional and viscosity solutions

In [7], H. Ishii showed that the two kinds of weak solutions of (1.7) we have been discussing are equivalent, under very nice regularity conditions on the coefficients as well as on the weak solution u . We state his results in our setting:

Theorem 5.7. ([7, Theorems 1 and 2]) *Let $\mathcal{Q} := (0, \infty) \times \mathcal{O}$. Assume that $a_{ij} \in C^{1,1}(\mathcal{Q})$ and $b_i \in C^{0,1}(\mathcal{Q})$ with \mathcal{Q} regarded as an open subset of $(0, \infty) \times \mathbb{R}^n$, and h and u are continuous. Then*

- (i) *if u is a viscosity solution of (1.7), it is also a distributional solution of (1.7); whereas*
- (ii) *if $\sigma \in C^1(\mathcal{Q})$ and u is a distributional solution of (1.7), then u is also a viscosity solution of (1.7).*

We have shown in Theorem 2.2 that \mathcal{U} is a viscosity solution of (1.7). In order to apply Theorem 5.7 to prove that \mathcal{U} is also a distributional solution, we need $a_{ij} \in C^{1,1}(\mathcal{Q})$, $b_i \in C^{0,1}(\mathcal{Q})$, and \mathcal{U} to be continuous. The difficulty in our setting comes from the fact that we want to assume the coefficients to be only continuous, and that the continuity of \mathcal{U} is not known *a priori*. However, if all of the conditions in Theorem 5.7 hold, we have shown that \mathcal{U} is then a classical solution (see the discussion in the second paragraph of Section 3).

6. Conclusions and future research

We have characterized the function \mathcal{U} as a viscosity solution of the Cauchy problem (1.7), (1.8), and shown that \mathcal{U} is dominated by any nonnegative classical supersolution of this Cauchy problem. We have also shown that \mathcal{U} is jointly continuous under appropriate conditions when $n = 1$, and therefore the smallest nonnegative classical (super)solution of this Cauchy problem. It would be interesting to investigate whether those conditions can be relaxed.

It would be also interesting to study whether \mathcal{U} is a distributional (super)solution, a requirement which consists of two parts: that $\mathcal{U} \in W_{loc}^{1,2}((0, \infty) \times \mathcal{O})$ and that (5.2) holds for some Γ . The analysis in Section 5.2 shows that one cannot apply the results or the approach in [7] directly.

Finally, it would be interesting to explore the joint continuity of \mathcal{U} in the multidimensional case; as was pointed out in the second paragraph of Section 3, with jointly continuous \mathcal{U} , continuous f , and locally Hölder $a (> 0)$, b and $h (\geq 0)$, one can conclude that \mathcal{U} is a classical solution to (1.7).

Appendix A: Another proof of Theorem 1.1 in the one-dimensional case

This section presents a proof of Theorem 1.1 in the one-dimensional case which is direct, that is, does not proceed by contradiction. This proof also works in several dimensions, whenever it is known in advance that the function U of (1.2) is continuous.

Suppose that $\mathcal{O} = (\ell, r)$ is a fixed open interval with $-\infty \leq \ell < r \leq \infty$.

Theorem A.1. *Assume that σ and b are continuous. Then the function U of (1.2) is a viscosity solution of the parabolic equation (1.3).*

Proof. Let us show that $U(t, x)$ is a viscosity subsolution to (1.3). The proof for the viscosity supersolution property is similar.

According to the definition of viscosity subsolution and the joint continuity of U ([9, Proposition 4.3]), it suffices to verify that for any $(t_0, x_0) \in (0, \infty) \times \mathcal{O}$ and any $\varphi(t, x) \in C^{1,2}((0, \infty) \times \mathcal{O})$ with

$$(U - \varphi)(t_0, x_0) = \max_{(t,x) \in (0,\infty) \times \mathcal{O}} (U - \varphi)(t, x) = 0, \quad (\text{A.1})$$

we have the inequality

$$G(t_0, x_0) \leq 0 \quad \text{for the function} \quad G(t, x) := (\varphi_t - \mathcal{L}\varphi)(t, x)$$

with the operator \mathcal{L} defined in (1.11).

We start the diffusion process $X(\cdot)$ at $X(0) = x_0$, consider $\alpha > 0$ such that $[x_0 - \alpha, x_0 + \alpha] \subset \mathcal{O}$, and define $T_\alpha := \inf \{t > 0 : |X(t) - x_0| > \alpha\}$. Then we have $S > T_\alpha$ if $T_\alpha < \infty$, and $S = \infty$ if $T_\alpha = \infty$. In either case, the bounded stopping time

$$\theta_m := \frac{1}{m} \wedge T_\alpha < S.$$

Further, for every fixed $\omega \in \Omega$, we notice that $\theta_m(\omega) = 1/m$ when $m > 1/T_\alpha(\omega)$.

Take $m > \frac{2}{t_0}$, then we have $\theta_m \leq \frac{1}{m} \wedge S \leq \frac{t_0}{2} \wedge S < t_0$. The strong Markov property of $X(\cdot)$ gives

$$\begin{aligned} \mathbb{E}_{x_0} [U(t_0 - \theta_m, X(\theta_m))] &= \mathbb{E}_{x_0} [\mathbb{P}_{X(\theta_m)}[S > t_0 - \theta_m]] = \mathbb{E}_{x_0} [\mathbb{P}_{x_0}[S > t_0 \mid \mathcal{F}(\theta_m)]] \\ &= \mathbb{P}_{x_0}[S > t_0] = U(t_0, x_0). \end{aligned}$$

On the other hand, we recall (A.1) and obtain

$$\varphi(t_0, x_0) - \mathbb{E}_{x_0} [\varphi(t_0 - \theta_m, X(\theta_m))] \leq U(t_0, x_0) - \mathbb{E}_{x_0} [U(t_0 - \theta_m, X(\theta_m))] = 0. \quad (\text{A.2})$$

Thanks to the assumption $\varphi \in C^{1,2}((0, \infty) \times \mathcal{O})$, we can apply Itô's change of variable rule to $\varphi(t_0 - s, X(s))$, $s \in [0, \theta_m] \subset [0, t_0]$ and plug in (1.1) to derive the semimartingale decomposition

$$d\varphi(t_0 - s, X(s)) = -G(t_0 - s, X(s)) ds + \varphi_x(t_0 - s, X(s)) \sigma(X(s)) dW(s). \quad (\text{A.3})$$

Integrating (A.3) with respect to s over $[0, \theta_m] \subset [0, t_0]$ and taking the expectation under \mathbb{P}_{x_0} yields

$$\begin{aligned} 0 &\geq m \cdot \text{LHS of (A.2)} = m \mathbb{E}_{x_0} \left[\int_0^{\theta_m} G(t_0 - s, X(s)) ds - \int_0^{\theta_m} \sigma(X(s)) \varphi_x(t_0 - s, X(s)) dW(s) \right] \\ &= \mathbb{E}_{x_0} \left[m \int_0^{\theta_m} G(t_0 - s, X(s)) ds \right] := G_m(t_0, x_0). \end{aligned}$$

Here in the second equality, the expectations of the integrals with respect to $dW_k(s)$ have all vanished, due to the local boundedness of functions σ and φ_x and the fact that for all $s \in [0, \theta_m]$, we have $t_0 - s \in [t_0/2, t_0]$ and $X(s) \in [x_0 - \alpha, x_0 + \alpha] \subset \mathcal{O}$ since $\theta_m < \frac{t_0}{2} \wedge S$.

It then suffices to show that $G_m(t_0, x_0) \rightarrow G(t_0, x_0)$ as $m \rightarrow \infty$. In fact, recalling that $\theta_m \leq 1/m$, as well as the boundedness of the terms in $G(t_0 - s, X(s))$, we can take the limit and appeal to the dominated convergence theorem to obtain

$$\lim_{m \rightarrow \infty} G_m(t_0, x_0) = \mathbb{E}_{x_0} \left[\lim_{m \rightarrow \infty} m \int_0^{\theta_m} G(t_0 - s, X(s)) ds \right].$$

Finally, since $\theta_m = 1/m \rightarrow 0$ for sufficiently large m , the mean value theorem and the continuity in s give $\lim_{m \rightarrow \infty} G_m(t_0, x_0) = G(t_0, x_0)$, as desired. \square

Appendix B: Proof of (2.14)

Proof. We have

$$\begin{aligned} \text{LHS} \cdot (Y(t))^{-1} &= -\varphi_t(t^* - t, X(t)) dt + \sum_i \varphi_{x_i}(t^* - t, X(t)) dX_i(t) \\ &\quad + \sum_{i,j} \frac{1}{2} \varphi_{x_i x_j}(t^* - t, X(t)) d\langle X_i, X_j \rangle(t) - h(X(t)) \varphi(t^* - t, X(t)) dt \\ &= -\varphi_t(t^* - t, X(t)) dt + \sum_{i,k} \varphi_{x_i}(t^* - t, X(t)) [\sigma_{ik}(X(t)) dW_k(t) + b_i(X(t)) dt] \\ &\quad + \sum_{i,j,k} \frac{1}{2} \varphi_{x_i x_j}(t^* - t, X(t)) \sigma_{ik}(X(t)) \sigma_{jk}(X(t)) dt - h(X(t)) \varphi(t^* - t, X(t)) dt \\ &= \left[-\varphi_t + \sum_i b_i(X(t)) \varphi_{x_i} + \sum_{i,j} \frac{1}{2} a_{ij}(X(t)) \varphi_{x_i x_j} - h(X(t)) \varphi \right] (t^* - t, X(t)) dt \\ &\quad + \sum_{i,k} \varphi_{x_i}(t^* - t, X(t)) \sigma_{ik}(X(t)) dW_k(t) \\ &= \text{RHS} \cdot (Y(t))^{-1}. \end{aligned} \quad \square$$

Acknowledgements

The author is greatly indebted to her advisor Professor Ioannis Karatzas for suggesting this problem and for his very careful reading of previous versions of the paper and invaluable advice. She is also indebted to Professor Johannes Ruf for his very careful reading of the manuscript and his many helpful suggestions. She is grateful to a referee, an area editor and the editor for their careful and helpful comments. This work was supported partly by the National Science Foundation under grant NSF-DMS-14-05210.

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